JOURNAL OF APPROXIMATION THEORY 26, 277–292 (1979)

A Global Inverse Theorem for Combinations of Bernstein Polynomials

Z. DITZIAN

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada T6G 2G1

Communicated by R. Bojanic

Received December 20, 1977

1. INTRODUCTION

The well-known Bernstein polynomials are given by

$$B_n(f, x) \equiv \sum_{k=0}^{n} \binom{n}{k} x^k(1-x)^{n-k} f\left(\frac{k}{n}\right) \equiv \sum_{k=0}^{n} f\left(\frac{k}{n}\right) p_{k,n}(x). \tag{1.1}$$

It was shown by H. Berens and G. G. Lorentz [1] that $|B_n(f, x) - f(x)| \leqslant M(x(1-x)/n)^{\alpha/2}$ if and only if $f \in \mathrm{Lip}^*\,\alpha$, i.e., if and only if $\|\Delta_h^2 f\|_{C[h,1-h]} \equiv \|f(x+h) - 2f(x) + f(x-h)\|_{C[h,1-h]} = O(h^\alpha)$. The combination of Bernstein polynomials given by

$$\begin{aligned} (2^r - 1)\, B_n(f, r, x) &\equiv 2^r B_{2n}(f, r-1, x) - B_n(f, r-1, x), \\ B_n(f, 0, x) &\equiv B_n(f, x). \end{aligned} \tag{1.2}$$

was introduced by P. L. Butzer [2] who showed that for smooth functions $B_n(f, k, x) - f(x)$ tends to zero faster than $B_n(f, x) - f(x)$. In [3] the local saturation of $B_n(f, k, x) - f(x)$ was investigated. C. P. May [5] found a local inverse theorem for $B_n(f, k, x) - f(x)$ (in fact, a more general combination was treated). These results related $\| B_n(f, k, \cdot) - f(\cdot)\|_{C[a,b]}$ to smoothness in (α, β) where $[\alpha, \beta] \subset (a, b)$ and $[a, b] \subset (0, 1)$. Since [3] and [5] have appeared some mathematicians expressed interest in the corresponding global result. In this paper we shall overcome the difficulty caused by the singularity at 0 and 1, and obtain such a result. We shall use the technique of space interpolation and characterize in Section 5 the intermediate space in an elementary way (that is, using smoothness rather than interpolation between spaces).

0021-9045/79/070277-16$02.00/0

640/26/3-7

2. The Main Result

In this section we shall state the main result of the paper. We shall also analyze what inequalities are needed for the proof which will be completed in later sections.

Let us denote $\|f\| = \sup_{0\leqslant x\leqslant 1} |f(x)|$, $\|f\|_{2r} = \sup_{0\leqslant x\leqslant 1} |f^{(2r)}(x)\, x^r(1-x)^r|$ and $A_{2r} = \{f: \|f\|_{2r} < \infty$ and $f^{(2r-1)}(x) \in AC$ locally$\}$ ($f^{(2r-1)}$ being absolutely continuous locally so that $\int^x f^{(2r)}(u)\, du = f^{(2r-1)}(x)$). The Peetre K functional is given by

$$K(t^{2r}, f) = \inf_{g\in A_{2r}} \{\|f-g\| + t^{2r}\|g\|_{2r}\}. \tag{2.1}$$

The intermediate space $(C, A_{2r})_\beta$ for some $0 < \beta < 2r$ is the collection of all f for which the norm $\sup_{t>0} t^{-\beta}K(t^{2r}, f)$ is finite. One can easily see that it is equivalent to use $K_1(t^{2r}, f) = \inf_{g\in A_{2r}}(\|f-g\| + t^{2r}(\|g\| + \|g\|_{2r}))$ instead of $K(t^{2r}, f)$.

Theorem 2.1. *For $f \in C[0, 1]$, $B_n(f, r-1, x)$ given by* (1.2) *and* $0 < \beta < 2r$ *the following are equivalent*:

(A) $\|B_n(f, r-1, x) - f(x)\| = O(n^{-\beta/2})$, $n \to \infty$;

(B) $f \in (C, A_{2r})_\beta$;

(C) $\sup_{hr<x<1-hr} |(1-x)^{\beta/2} h^{-\beta} \Delta_h^{2r} f(x)| < M$ *where* $\Delta_h f(x) = f(x+h/2) - f(x-h/2)$.

Theorem 2.2. *In Theorem* 2.1, (A) *can be replaced by*

$$\|B_{n(r)}(f, x) - f(x)\| = O(n^{-\beta/2}), \qquad n \to \infty$$

where

$$B_{n(r)}(f, k) \equiv \sum_{i=0}^{r-1} C_i(n)\, B_{n_i}(f, x)$$

satisfies:

(a) $n = n_0 < n_i < n_{r-1} < Kn$ (*K independent of n*);

(b) $\sum_{i=0}^{r-1} |C_i(n)| < C$ (*C independent of n*);

(c) $\sum_{i=0}^{r-1} C_i(n) = 1$; *and*

(d) $\sum_{i=0}^{r-1} C_i(n)(1/n_i^\rho) = 0$, $\rho = 1, 2,\ldots, r-1$.

Remark 2.1. It is obvious that $B_n(f, r-1, x)$ is a special case of $B_{n(r)}(f, x)$ with $C_i(n) = C_i$ and $n_i = 2^i n$. The combinations treated by C. P. May [5] are also a special case. However, we do not see much advantage in the generalization of Theorem 2.1 in Theorem 2.2 as it yields no new idea but is just an observation of what is used. We will simply mention, while proving Theorem 2.1, when (a)–(d) may replace special properties of $B_n(f, r-1, x)$.

Remark 2.2. For $r = 1$ the equivalence of (A) and (B) as well as (B) implies (C) was shown by Berens and Lorentz [1]. However, (C) implies (B) was not shown there and is new even for $r = 1$.

Remark 2.3. Since the result displayed most prominantly in Berens and Lorentz' paper is $|B_n(f, x) - f(x)| \leqslant M(x(1-x)/n)^{\alpha/2}$ if and only if $f \in \mathrm{Lip}^* \alpha$, $0 < \alpha < 2$, one would expect $|B_n(f, r-1, x) - f(x)| \leqslant M(x(1-x)/n)^{\alpha/2}$ if and only if $\sup_{rh<x<1-rh} |\Delta_h^{2r} f(x)| \leqslant Mh^\alpha$ for $0 < \alpha < 2r$. This result is not true since for $f(x) = x^3$ and $r = 2$ $|B_n(f, 1, x) - f(x)| \sim x(1-x)(1-2x)/n^2$ (and one may choose $x = 1/n$ and $\alpha = 4 - \epsilon$).

The proof of the equivalence of (B) and (C) has no relation with Bernstein polynomials and will be given in Section 5.

The proof of the implication (B) $\Rightarrow$ (A) is of the type usually called direct theorem and will be obtained in Section 4.

To prove that (A) implies (B), a direction usually called the inverse result, we write

$$K(t^{2r}, f) \leqslant \| B_n(f, r-1, x) - f(x)\| + t^{2r} \| B_n(f, r-1, x)\|_{2r}. \quad (2.2)$$

We recall that (A) yields $\| B_n(f, r-1, x) - f(x)\| \leqslant M/n^{\beta/2}$ and that (1.2) yields $B_n(f, r-1, x) = \sum_{0 \leqslant i < r-1} C_i B_{2^i n}(f, x)$ and therefore

$$\| B_n(f, r-1, x)\|_{2r} \leqslant M_1 \sup_{0 \leqslant i < r-1} \| B_{2^i n}(f, x)\|_{2r}. \quad (2.3)$$

It is enough to show the following two inequalities:

$$\| B_n(f, x)\|_{2r} \leqslant M_2 n^r \|f\| \quad (2.4)$$

and

$$\| B_n(f, x)\|_{2r} \leqslant M_3 \|f\|_{2r} \qquad \text{for} \quad f \in A_{2r}. \quad (2.5)$$

These combined with (2.2) and (2.3) imply

$$K(t^{2r}, f) \leqslant M_4(n^{-\beta/2} + t^{2r} n^r K(n^{-r}, f)), \qquad n \geqslant n_0, \quad (2.6)$$

which, following Berens and Lorentz' method, implies $K(t^{2r}, f) \leqslant M_5 t^\beta$. We shall prove (2.4) and (2.5) in Section 3. We may observe that, pending (2.4) and (2.5), we proved that $\| \sum_{i=0}^{r-1} C_i(n) B_{n_i}(f, x) - f(x)\| \leqslant M_1 n^{-\beta/2}$ with $|C_i(n)| \leqslant M_2$ and $n = n_0 < n_1 < \cdots < n_{r-1} \leqslant Kn$ implies $f \in (C, A_{2r})_\beta$ and this does not depend on the particular structure of (1.2). It is for the direct theorem that we need either (1.2) or a similar combinatorial structure ((c) and (d) of Theorem 2.2).

3. The Completion of the Inverse Result

In this section we shall prove (2.4) and (2.5) which together will complete the proof of the inverse result.

Lemma 3.1. *For* $f \in C[0, 1]$,

$$B_n^{(l)}(f, x) = \frac{n!}{(n-l)!} \sum_{k=0}^{n-l} \Delta_{1/n}^{l} f\left(\frac{k}{n}\right) P_{n-l,k}(x)$$

where (3.1)

$$\Delta_{1/n} f(x) = f\left(x + \frac{1}{n}\right) - f(x).$$

Proof. See [4, p. 12]. ($\Delta_{1/n}^{l} f(x)$ here is not the same as in [C] but rather as in [4]; it will be used only in this section, so there is no possibility of confusion).

Lemma 3.2. *For* $P_{n,k}(x)$ *given in* (1.1) *we have*

$$\sum_{k=0}^{n} \frac{n^m}{(k+1)^m} P_{n,k}(x) \leqslant \frac{m!}{x^m} \tag{3.2}$$

and

$$\sum_{k=0}^{n} \frac{n^m}{(n-k+1)^m} P_{n,k}(x) \leqslant \frac{m!}{(1-x)^m}. \tag{3.3}$$

Proof. Inequality (3.3) follows (3.2) substituting $\xi = 1 - x$ and $\nu = n - k$. We can write

$$\sum_{k=0}^{n} \frac{n^m}{(k+1)^m} P_{n,k}(x) \leqslant \frac{1}{x^m} \sum_{k=0}^{n} \frac{n^m}{(k+1)^m} \frac{(k+1)\cdots(k+m)}{(n+1)\cdots(n+m)} P_{n+m,k+m}(x)$$

$$\leqslant \frac{m!}{x^m} \sum_{k=0}^{n} P_{n+m,k+m}(x) \leqslant \frac{m!}{x^m} \sum_{v=0}^{n+m} P_{n+m,v}(x) \leqslant \frac{m!}{x^m}.$$

Lemma 3.3. *Suppose* $X^r f^{(2r)}(x) \in L_\infty[0, 1]$, *and* $f^{(2r-1)}(x) \in A.C.[\alpha, \beta]$ *for all* α, β, $0 < \alpha < \beta < 1$, *where* $X = x(1-x)$. *Then, for* $m = 1, 2, \ldots, r-1$,

$$|X^{r-m} f^{(2r-m)}(x)| \leqslant B(m)(\|f\|_{2r} + \|f\|). \tag{3.4}$$

Proof. It is well known that

$$|f^{(2r-m)}(\tfrac{1}{2})| \leqslant B_m(\|f^{(2r)}\|_{L_\infty(1/4,3/4)} + \|f\|_{C(1/4,3/4)}) \leqslant KB_m(\|f\|_{2r} + \|f\|),$$

since this follows a Kolmogorov-type inequality for an interval. It is enough to prove the theorem for $x \leqslant \frac{1}{2}$ since the proof for $x \geqslant \frac{1}{2}$ is very similar. By induction we have

$$|f^{(2r-m)}(x) - f^{(2r-m)}(\tfrac{1}{2})| = \left| \int_x^{1/2} f^{(2r-m+1)}(u)\, du \right|$$
$$\leqslant B(m-1) \cdot K \cdot (\|f\|_{2r} + \|f\|) \int_x^{1/2} \frac{du}{u^{r-m+1}}$$

and therefore

$$x^{r-m} | f^{(2r-m)}(x)| \leqslant B(m-1)\, K(\|f\|_{2r} + \|f\|) + | f^{(2r-m)}(\tfrac{1}{2})|$$

which completes the proof.

We are now in a position to prove (2.5) and (2.4) which will constitute Lemmas 3.4 and 3.5 respectively.

LEMMA 3.4. *Let $f \in A_{2r}$; then* (2.5) *is valid, i.e.*

$$\| B_n(f, x)\|_{2r} \leqslant K \|f\|_{2r}.$$

Proof. For $1 \leqslant k \leqslant n - 2r - 1$ and $\Delta_{1/n} f(k/n) = f((k+1)/n) - f(k/n)$, we have

$$n^{2r} \left| \Delta_{1/n}^{2r} f\left(\frac{k}{n}\right)\right| \leqslant \sup_{(k/n)<\xi<(k+2r/n)} | f^{(2r)}(\xi)| \leqslant \frac{\|f\|_{2r}}{(k/n)^r\, ((n-k-2r)/n)^r}.$$

We will show that for $k = 0$ (and similarly for $k = n - 2r$) $| n^{2r}\Delta_{1/n}^{2r} f(0)| < Kn^r \|f\|_{2r}$. Using Lemma 3.3 (which will be used later in full generality) to justify integration by parts, we have

$$n^{2r}\Delta_{1/n}^{2r} f(0) = n^{2r} \sum_{k=0}^{2r} \binom{2r}{k}(-1)^k f\left(\frac{k}{n}\right)$$
$$= \frac{n^{2r}}{(2r-1)!} \sum_{k=1}^{2r} \binom{2r}{k}(-1)^k \int_0^{k/n} u^{2r-1} f^{(2r)}(u)\, du$$
$$+ \sum_{i=1}^{2r-1} \frac{n^{2r}(-1)^{i+1}}{i!} \sum_{k=1}^{2r} (-1)^k \binom{2r}{k}\left(\frac{k}{n}\right)^i f^{(i)}\left(\frac{k}{n}\right) \equiv I + \sum_{i=1}^{2r-1} J_i.$$

We can now write

$$| I | \leqslant \frac{n^{2r}}{(2r-1)!}\, 2^{2r} \int_0^{2r/n} u^{2r-1}\, | f^{(2r)}(u)|\, du \leqslant M_1 \|f\|_{2r}\, n^r;$$

and

$$\begin{aligned}
J_i &= \frac{n^{2r-i}}{i!}(-1)^{i+1}\sum_{k=1}^{2r}(-1)^k\binom{2r}{k}k^i f^{(i)}\left(\frac{k}{n}\right)\\
&= \frac{n^{2r-i}}{i!}(-1)^{i+1}\sum_{k=1}^{2r}(-1)^k\binom{2r}{k}\left(\sum_{l=0}^{i-1}\alpha_l k(k-1)\cdots(k-l)\right)f^{(i)}\left(\frac{k}{n}\right)\\
&\equiv \sum_{l=0}^{i-1} J_{i,l}
\end{aligned}$$

where α_l depends on l and i only. We have now

$$\begin{aligned}
|J_{i,l}| &\leqslant \left|\alpha_l \frac{2r\cdots(2r-l)}{i!}n^{2r-i}\sum_{k=l+1}^{2r}(-1)^k\binom{2r-l-1}{k-l-1}f^{(i)}\left(\frac{k}{n}\right)\right|\\
&= \left|\alpha_l \frac{2r\cdots(2r-l)}{i!}n^{2r-i}\Delta_{1/n}^{2r-l-1}f^{(i)}\left(\frac{l+1}{n}\right)\right|\\
&\leqslant M_{i,l}n^{2r-i}\sup_{l\leqslant j\leqslant i-1}\left|\Delta_{1/n}^{2r-i}f^{(i)}\left(\frac{j+1}{n}\right)\right|\\
&\leqslant M_{i,l}\sup_{(l+1)/n\leqslant\xi\leqslant 2r/n}|f^{(2r)}(\xi)| \leqslant M_{i,l}\frac{\|f\|_{2r}}{((l+1)/n)^r}
\end{aligned}$$

which concludes the proof of $|n^{2r}\Delta_{1/n}^{2r}f(0)| \leqslant Kn^r\|f\|_{2r}$.

Therefore, using Lemma 3.1,

$$\begin{aligned}
&X^r\,|B_n^{(2r)}(f,x)|\\
&\quad\leqslant n^{2r}X^r\sum_{k=0}^{n-2r}\left|\Delta_{1/n}^{2r}f\left(\frac{k}{n}\right)\right| P_{n-2r,k}(x)\\
&\quad\leqslant K_1\|f\|_{2r}X^r\sum_{k=0}^{n-2r}\frac{P_{n-2r,k}(x)}{((k+1)/n)^r((n-2r+1-k)/n)^r}\\
&\quad\leqslant K_1\|f\|_{2r}\sum_{k=0}^{n-2r}\frac{X^r}{((k+1)/n)^r}+\frac{X^r}{((n-2r+1-k)/n)^r}P_{n-2r,k}(x)
\end{aligned}$$

which, using Lemma 3.2, implies (2.5).

LEMMA 3.5. *For* $f\in C[0,1]$, (2.4) *is valid, i.e.*, $\|B_n(f,x)\|_{2r}\leqslant An^r\|f\|$.

Proof. For x that satisfies $\min(x, 1 - x) < B/n$, using Lemma 3.1, we write

$$X^r \mid B_n^{(2r)}(f, x)| \leqslant X^r n \cdots (n - 2r + 1) \sum_{k=0}^{n-2r} \left| \Delta_{1/n}^{2r} f\left(\frac{k}{n}\right)\right| P_{n-2r,k}(x)$$
$$\leqslant (x(1 - x))^r n^{2r} 2^r \|f\| \leqslant n^r \cdot 2^r \|f\| [nx(1 - x)]^r$$
$$\leqslant n^r 2^r B^r \|f\|.$$

The case $B/n < x < 1 - B/n$ is somewhat more complicated. One can write $P'_{n,k}(x) = ((k - nx)/x(1 - x)) P_{n,k}(x)$. Repeated differentiation shows that $P_{n,k}^{(2r)}(x)$ is a sum of terms of the type

$$q_{l,m}(x) \frac{(k - nx)^{2r-2l-m} n^l}{(x(1 - x))^{2r-l}} P_{n,k}(x)$$

where $l \geqslant 0$, $m \geqslant 0$, $2r - 2l - m \geqslant 0$ and $q_{l,m}(x)$ is a polynomial in x that does not depend on n and k. To complete the proof of the Lemma we have to estimate $| X^r B_n^{(2r)}(f, x)|$ for $B/n \leqslant x \leqslant 1 - (B/n)(B/n < \frac{1}{2})$. Since $X^r B_n^{(2r)}(f, x)$ is the sum of elements of the type

$$I(l, m, n) = q_{l,m}(x) X^{l-r} n^l \sum_{k=0}^{n} (k - nx)^{2r-2l-m} f\left(\frac{k}{n}\right) P_{n,k}(x) \text{ for } l, m \geqslant 0$$

and $2r - 2l - m \geqslant 0$, we have to estimate $I(l, m, n)$ for $B/n < x < 1 - B/n$. Using the Cauchy–Schwartz inequality, we have

$$| I(l, m, n)| \leqslant | q_{l,m}(x)| X^{l-r} n^l \|f\| \left(\sum_{k=0}^{n} | k - nx |^{4r-4l-2m} P_{n,k}(x)\right)^{1/2}.$$

Recalling the definition of $T_{n,s}(x)$ ([2] and [4]) i.e.

$$T_{n,s}(x) = \sum_{k=0}^{n} (k - nx)^s P_{n,k}(x), \tag{3.5}$$

we shall show

$$| T_{n,2s}(x)| \leqslant K n^s X^s \text{ for } \frac{B}{n} \leqslant x \leqslant 1 - \frac{B}{n}. \tag{3.6}$$

Using (3.6), we have

$$I(l, m, n) \leqslant K X^{l-r} n^l n^{r-l-m/2} X^{r-l-m/2} \|f\|$$
$$\leqslant K n^r [nx(1 - x)]^{-m/2} \|f\| \leqslant K \left(\frac{B}{2}\right)^{-m/2} n^r \|f\|.$$

To prove estimate (3.6) we recall [2, p. 56]

$$T_{n,s+1}(x) = X[T'_{n,s}(x) + nsT_{n,s-1}(x)],\ T_{n,0}(x) = 1,\ T_{n,1}(x) = 0. \quad (3.7)$$

Therefore (as can be proved by induction),

$$T_{n,2s}(x) = p_s(n(x(1-x)))^s + \sum_{i=1}^{s-1} p_{i,s}(x)[nx(1-x)]^{s-i} \quad (3.8)$$

and

$$T_{n,2s+1} = b_s(1-2x)(n(x(1-x)))^s + \sum_{i=1}^{s-1} b_{i,s}(x)[nx(1-x)]^{s-i} \quad (3.9)$$

where $p_{i,s}(x)$ and $b_{i,s}(x)$ are independent of n. Since $nx(1-x) > B/2$,

$$\begin{aligned} |T_{n,s}(x)| &\leqslant |p_s|\, n(x(1-x))^s + \left(\sum_{i=1}^{s-1} |p_{i,s}(x)| \left(\frac{2}{B}\right)^i\right)(nx(1-x))^s \\ &\leqslant K(nx(1-x))^s \end{aligned}$$

which completes the proof of (3.6) and therefore of Lemma 3.5.

4. The Direct Theorem

The direct part of Theorem 2.1 is the following:

Direct Theorem. *Let* $f \in (C, A_{2r})_\beta$, *then* $\| B_n(f, r-1, x) - f(x)\| \leqslant M/n^{\beta/2}$.

Proof. For $f \in (C, A_{2r})_\beta$ and any t there exist $g_t \in A_{2r}$ such that $\| f - g_t \| \leqslant M_1 t^\beta$ and $t^{2r} \| g_t \|_{2r} \leqslant M_1 t^\beta$ or $\| g_t \|_{2r} \leqslant M_1 t^{\beta - 2r}$. Therefore, recalling $\| B_n(\phi, r-1, x)\| \leqslant M_2 \| \phi \|$, we have

$$\begin{aligned} &\| B_n(f, r-1, x) - f(x)\| \\ &\quad\leqslant \| B_n(f - g_t, r-1, x)\| + \| f - g_t \| + \| B_n(g_t, r-1, x) - g_t(x)\| \\ &\quad\leqslant (M_2 + 1) M_1 t^\beta + \| B_n(g_t, r-1, x) - g_t(x)\|. \end{aligned}$$

Choosing $t = n^{-1/2}$, we have only to show

$$\| B_n(\phi, r-1, x) - \phi(x)\| \leqslant M_3 n^{-r}(\| \phi \|_{2r} + \| \phi \|) \qquad \text{for} \quad \phi \in A_{2r}. \quad (4.1)$$

Inequality (4.1) is the crucial step in the direct theorem and the only step in which the combination (1.2) (or a similar condition) need be considered.

One should also note that actually (4.1) is the direct part of the global saturation result.

For $A/n \leqslant x \leqslant 1 - A/n$ we will use Taylor's formula around x and write

$$
\begin{aligned}
&B_n(\phi, r-1, x) - \phi(x) \\
&\quad = \sum_{l=1}^{2r-1} \sum_{i=0}^{r-1} C_i \sum_{k=0}^{i} \frac{1}{l!} \left(\frac{k}{2^i n} - x\right)^l P_{2^i n,k}(x)\, \phi^{(l)}(x) \\
&\quad\quad - \frac{1}{(2r-1)!} \sum_{i=0}^{r-1} C_i \sum_{k=0}^{2^i n} P_{2^i n,k}(x) \int_{k/2^i n}^{x} \left(\frac{k}{2^i n} - u\right)^{2r-1} \phi^{(2r)}(u)\, du \\
&\quad = I_1 + I_2 .
\end{aligned}
$$

Formulae (3.8) and (3.9) can also be written as

$$
\begin{aligned}
&\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^{2l} P_{n.k}(x) \\
&\quad = \frac{1}{n^l} \left(A_1(x(1-x))^l + A_2(x) \frac{(x(1-x))^{l-1}}{n} + \cdots + A_l(x) \frac{x(1-x)}{n^{l-1}}\right)
\end{aligned}
\tag{4.2}
$$

and

$$
\begin{aligned}
\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^{2l+1} P_{n.k}(x) = \frac{1}{n^{l+1}} \Big(&B_1(1-2x)(x(1-x))^l \\
&+ B_2(x) \frac{(x(1-x))^{l-1}}{n} + \cdots + B_l(x) \frac{x(1-x)}{n^{l-1}}\Big) .
\end{aligned}
\tag{4.3}
$$

To estimate I_1 we simply recall

$$
\sum_{i=0}^{r-1} C_i \frac{1}{2^{il}} = 0 \qquad \text{for} \quad l = 1, 2, \ldots, r-1 \tag{4.4}
$$

and therefore, using (4.2) and (4.3), many terms are eliminated and we have in I_1 only terms which are polynomials multiplied by

(a) $J_1(j, l) = (1/n^{l+j})(x(1-x))^{l-j}\, \phi^{(2l)}(x)$ for $l + j \geqslant r$ and $2l > r$ or

(b) $J_2(j, l) = (1/n^{l+j+1})(x(1-x))^{l-j}\, \phi^{(2l+1)}(x)$ for $l + j + 1 \geqslant r$ and $2l + 1 > r$.

In case (a) $l + j = r$ or (b) $l + j + 1 = r$, using (3.4) of Lemma (3.3) with $m = 2r - 2l$ or $m = 2r - 2l - 1$ respectively, we have the estimate $| J_i(j, l) \leqslant K/n^r(\| \phi \|_{2r} + \| \phi \|)$, $i = 1, 2$. For bigger r (than in (a) or (b)) we note $nx(1-x) \geqslant A/2$, since we assumed $A/n < x < 1 - A/n$ for this part of the proof, and therefore, we may multiply by $(2nx(1-x)/A)$ until the denominator is exactly $1/n^r$.

To estimate I_2 for $A/n < x < 1 - A/n$ we write

$$\begin{aligned}
| I_2 | &\leqslant K \sum_{k=0}^{n} P_{n,k}(x) \left| \int_{k/n}^{x} \left(\frac{k}{n} - u\right)^{2r-1} \phi^{(2r)}(u)\, du \right| \\
&= K \sum_{k=0}^{n} P_{n,k}(x) \| \phi \|_{2r} \left| \int_{k/n}^{x} \frac{| k/n - u |^{2r-1}}{(u(1-u))^r}\, du \right| \\
&\leqslant K \| \phi \|_{2r} \left(\sum_{k=0}^{n} P_{n,k}(x) (x(1-x))^{-r} \left(\frac{k}{n} - x\right)^{2r} \right. \\
&\quad \left. + \sum_{k=1}^{n-1} P_{n,k}(x) \left(\frac{k}{n} - x\right)^{2r} \left(\frac{k}{n}\left(1 - \frac{k}{n}\right)\right)^{-r} \right) \\
&\equiv I_2(1) + I_2(2).
\end{aligned}$$

Using the Cauchy–Schwartz inequality, Lemma 3.2 and (4.2) with $A/n < x < 1 - A/n$, we have

$$\begin{aligned}
I_2(2) &\leqslant K \| \phi \|_{2r} \left(\sum_{k=1}^{n-1} P_{n,k}(x) \left(\frac{k}{n} - x\right)^{4r} \right)^{1/2} \left(\sum_{k=1}^{n-1} P_{n,k}(x) \left(\frac{k}{n}\left(1 - \frac{k}{n}\right)\right)^{-2r} \right)^{1/2} \\
&\leqslant K_2 \| \phi \|_{2r}\, n^{-r} X^r X^{-r} \leqslant K_2 \| \phi \|_{2r}\, n^{-r}.
\end{aligned}$$

The estimate of $I_2(1)$ is similar to that of I_1.

For $\min(x, 1-x) < A/n$ we use Taylor's formula around x and write

$$\begin{aligned}
B_n(\phi, r-1, x) - \phi(x) &= \sum_{l=1}^{r} \frac{1}{l!} \sum_{i=0}^{r-1} C_i \sum_{k=0}^{2^i n} \left(\frac{k}{2^i n} - x\right)^l P_{2^i n,k}(x)\, \phi^{(l)}(x) \\
&\quad - \frac{1}{(r+1)!} \sum_{i=0}^{r-1} C_i \sum_{k=0}^{2^i n} P_{2^i n,k}(x) \\
&\quad \times \int_{k/2^i n}^{x} \left(\frac{k}{2^i n} - u\right)^r \phi^{(r+1)}(u)\, du \\
&= I_1 + I_2 .
\end{aligned}$$

Using (4.2) and (4.3), we observe that for I_1, since $l = 0, 1,..., r-1$, the coefficients of $\phi^{(l)}(x)$ are polynomials multiplied by $1/n^j$, $j \leqslant r-1$, and therefore, using (4.4), $I_1 = 0$. Using Lemma 3.3, we have

$$| U\phi^{(r+1)}(u)| \leqslant K(\| \phi \|_{2r} + \| \phi \|)$$

and since for $A \leqslant 1/2^{r+1}$, $\min(x, 1-x) < |x - k/2^i n|$, we have

$$| I_2 | \leqslant K(\| \phi \|_{2r} + \| \phi \|) \sum_{i=0}^{r-1} | C_i | \sum_{k=0}^{2^i n} \left| \frac{k}{2^i n} - x \right|^r P_{2^i n,k}(x)$$

$$\leqslant KC(\| \phi \|_{2r} + \| \phi \|) \sum_{i=0}^{r-1} | C_i | \left(\sum_{k=0}^{2^i n} \left| \frac{k}{2^{in}} - x \right|^{2r} P_{2^i n,k}(x) \right)^{1/2}.$$

The condition $\min(x, 1-x) < A/n$ implies $A/nx(1-x) > 1$, and multiplying by $(A/nx(1-x))^m$ wherever $(x(1-x))^m$ appears in (4.2) for $l = r$, we have $| \sum_{k=0}^{2^i n} |(k/2^i n) - x|^{2r} P_{2^i n,k}(x)| \leqslant K_i/n^{2r}$ and therefore $I_2 \leqslant M(\| \phi \|_{2r} + \| \phi \|)(1/n^r)$.

5. The Elementary Description of the Interpolation Space

In this section we shall prove the equivalence of (C) and (A) or (B) of Theorem 2.1 which will complete the proof of that theorem. Since the result is independent of the Bernstein polynomial approximation, we will summarize it separately by:

THEOREM 5.1. *For $f \in C[0, 1]$ the conditions*

$$\sup_{rh<x<1-rh} | X^{\beta/2} h^{-\beta} \Delta_h^{2r} f(x)| \leqslant M$$

$$\left(\text{where } X = x(1-x) \text{ and } \Delta f(x) = f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right)\right) \quad (5.1)$$

and $f \in (C, A_{2r})_\beta$ are equivalent.

Recall that $A_{2r} = \{f : f^{(2r-1)} \in A.C.$ locally in $(0, 1)$ and $| X^r f^{(2r)}(x)| < M\}$ and that for $f \in (C, A_{2r})_\beta$ there exists M_1 such that for every t there exists $g_t \in A_{2r}$ satisfying

$$\| f - g_t \| + t^{2r} \| g_t \|_{2r} \leqslant M_1 t^\beta \text{ or } \| f - g_t \| \leqslant M_1 t^\beta \text{ and } \| g_t \|_{2r} \leqslant M_1 t^{\beta - 2r}.$$

Therefore,

$$| X^{\beta/2} h^{-\beta} \Delta_h^{2r} f(x)| \leqslant X^{\beta/2} h^{-\beta} | \Delta_h^{2r}(f(x) - g_t(x))| + X^{\beta/2} h^{-\beta} | \Delta_h^{2r} g_t(x)|$$

$$\leqslant X^{\beta/2} h^{-\beta} 2^{2r} \| f - g_t \| + X^{\beta/2} h^{-\beta} | \Delta_h^{2r} g_t(x)| .$$

Choosing $t = h \cdot X^{-1/2}$, it is clear that we can complete the proof that $f \in (C, A_{2r})_\beta$ satisfies (5.1) if we can show for all $g_t \in A_{2r}$

$$|\Delta_h^{2r} g_t(x)| \leqslant Kh^{2r}X^{-r} \| g_t \|_{2r} \text{ where } K \text{ is independent of}$$
$$h < \frac{1}{2r}, \ X \text{ and } g_t. \tag{5.2}$$

To prove (5.2) we use the integral form of Taylor's formula

$$|\Delta_h^{2r} g_t(x)| \leqslant \sum_{l \neq 0, l=-r}^{r} \binom{2r}{r+l} \left| \int_{x-lh}^{x} (x - lh - u)^{2r-1} g_t^{(2r)}(u)\, du \right|.$$

Using the definition of $\| g_t \|_{2r}$, we have

$$\int_{x-lh}^{x} (u - x + lh)^{2r-1} | g_t^{(2r)}(u)|\, du \leqslant \| g_t \|_{2r} \int_{x-lh}^{x} \frac{(u - x + lh)^{2r-1}}{U^r}\, du.$$

For $x \leqslant \frac{1}{2}$ and $h < 1/4r$ (and therefore $1 - u > \frac{1}{4}$ for $x \leqslant u \leqslant x + rh$) and $l > 0$ we have

$$\begin{aligned} I &= \int_{x-lh}^{x} | x - lh - u |^{2r-1} U^{-r}\, du \leqslant 4^r \int_{x-lh}^{x} | x - lh - u |^{2r-1} u^{-r}\, du \\ &\leqslant 4^r \min \left\{ \int_{x-lh}^{x} | x - lh - u |^{r-1}\, du, \frac{1}{(x - lh)^r} \int_{x-lh}^{x} | x - lh - u |^{2r-1}\, du \right\} \\ &\leqslant 4^r \min \left(\frac{1}{r} (lh)^r, \frac{1}{(x - lh)^r} \frac{1}{2r} (lh)^{2r} \right). \end{aligned}$$

Using the estimate $(1/r)(lh)^r$ and $(1/(x - lh)^r)(1/2r)(lh)^{2r}$ for $x \leqslant (r + 1)h$ and $x > (r + 1)h$ respectively, we have

$$I \leqslant 4^r \frac{1}{r} (lh)^r \leqslant 4^r \frac{l^r}{r} \frac{h^{2r}}{h^r} \leqslant M \frac{h^{2r}}{X^r}$$

and

$$I \leqslant 4^r \frac{1}{2r} \frac{(lh)^{2r}}{(x - lh)^r} \leqslant 4^r \frac{l^{2r}}{2r} \left(\frac{x}{x - lh} \right)^r \frac{h^{2r}}{x^r} \leqslant M \frac{h^{2r}}{X^r}.$$

For $l < 0$ the estimate is somewhat simpler and for $x \geqslant \frac{1}{2}$ a similar estimate holds.

To prove that (5.1) implies $f \in (C, A_{2r})_\beta$ we have to construct g_t such that $\| f - g_t \| \leqslant M_1 t^\beta$ and $\| g_t \|_{2r} \leqslant M_1 t^{\beta - 2r}$. Let $\psi(x)$ be a $C^{2r}(I)$ function satisfying $\psi(x) = 1$ for $x \leqslant \frac{1}{4}$, $\psi(x) = 0$ for $x \geqslant \frac{3}{4}$ and $\psi'(x) \leqslant 0$. Obviously $f(x) = f(x)\,\psi(x) + f(x)(1 - \psi(x)) \equiv f_1(x) + f_2(x)$, and it is enough to find $g_{i,t}$ such that $\| g_{it} - f_i \| \leqslant (M_1/2)\, t^\beta$ and $\| g_{it} \|_{2r} \leqslant (M_1/2)\, t^{-2r+\beta}$. We shall construct $g_{1t}(x)$ in detail, and $g_{2t}(x)$ can be given similarly. Let $\psi_l(x) \equiv \psi(4^l x)$. We can now write

$$f_1(x) = f(x)\,\psi(x) = \sum_{k=0}^{l-1} f(x)\,\psi_k(x)(1 - \psi_{k+1}(x)) + f(x)\,\psi_l(x) \tag{5.3}$$

because $\operatorname{supp} \psi_{k+1}(x) \subset \{x\colon \psi_k(x) = 1\}$.

Define for $h \geqslant 0$

$$f_h(x) = \left(\frac{2r}{h}\right)^{2r} \int_0^{h/2r} \cdots \int_0^{h/2r} \sum_{j=1}^{2r} \binom{2r}{j} (-1)^{j+1} f(x + j(u_1 + \cdots + u_{2r})) \times du_1 \cdots du_{2r}\,. \tag{5.4}$$

Using (5.1) for $x < \frac{3}{4}$ and $h < 1/16r$ (and therefore $1 - (x + 2rh) > \frac{1}{8}$),

$$\begin{aligned} |f(x) - f_h(x)| &\leqslant 8^{\beta/2} \left(\frac{2r}{h}\right)^{2r} \int_0^{h/2r} \cdots \int_0^{h/2r} \frac{M(u_1 + \cdots + u_{2r})^\beta}{(x + r(u_1 + \cdots + u_{2r}))^{\beta/2}}\, du_1 \cdots du_{2r} \\ &\leqslant \min \left\{ M_2 \frac{h^\beta}{x^{\beta/2}}\,,\ M_2 \frac{1}{r^{\beta/2}}\, h^{\beta/2} \right\}. \end{aligned} \tag{5.5}$$

Moreover

$$\begin{aligned} |f_h^{(2r)}(x)| &= \left| \left(\frac{2r}{h}\right)^{2r} \sum_{j=1}^{2r} \binom{2r}{j} (-1)^{j+1} \Delta_{jh/2r}^{2r} f(x + \tfrac{1}{2} jh) \right| \\ &\leqslant \left(\frac{2r}{h}\right)^{2r} \sum_{j=1}^{2r} \binom{2r}{j} \frac{M_1 h^\beta 8^{\beta/2}}{(x + jh/2)^{\beta/2}} \leqslant M_2 h^{-2r} \frac{h^\beta}{(x + h/2)^{\beta/2}} \\ &\leqslant M_3 h^{-2r} \min \left(\frac{h^\beta}{x^{\beta/2}}\,,\ h^{\beta/2} \right). \end{aligned}$$

We define for $t < 1/16r$ and l such that $2^{-l-1} < t \leqslant 2^{-l}$

$$g_{1,t}(x) = \sum_{k=0}^{l-1} f_{t\cdot 2^{-k}}(x)\,\psi_k(x)(1 - \psi_{k+1}(x)) + f_{t\cdot 2^{-l}}(x)\,\psi_l(x). \tag{5.7}$$

For every x at most two terms in (5.3) and (5.7) are different from 0. In fact for $4^{-m-1} < x < 3 \cdot 4^{-m-1}$, $m \leqslant l$ both $\psi_m(x)(1 - \psi_{m+1}(x)) = \psi_m(x)$ and $\psi_{m-1}(x)(1 - \psi_m(x)) = 1 - \psi_m(x)$ may be different from 0 and for $3 \cdot 4^{-m-1} \leqslant x \leqslant 4^{-m}$ and $x \leqslant 4^{-l-1}$ $(1 - \psi_m(x))\,\psi_{m-1}(x) = 1$ and $\psi_l(x) = 1$ respectively, and therefore all other terms are equal to 0.

Combining (5.3), (5.4), (5.5) and (5.7) we have for $4^{-m-1} < x < 3 \cdot 4^{-m-1}$

$$|g_{1t}(x) - f_1(x)| \leqslant M_2 \frac{(t \cdot 2^{-m})^\beta}{(4^{-m})^{\beta/2}} + M_2 \frac{(t \cdot 2^{-m+1})^\beta}{(4^{-m})^{\beta/2}} \leqslant M_2(1 + 2^\beta)\, t^\beta,$$

for $3 \cdot 4^{-m-1} \leqslant x \leqslant 4^{-m}$

$$| g_{1t}(x) - f_1(x)| \leqslant M_2 \frac{(t \cdot 2^{-m+1})^\beta}{(3 \cdot 4^{-m})^{\beta/2}} \leqslant M_3 t^\beta$$

and for $x \leqslant 4^{-l-1}$

$$| g_{1t}(x) - f_1(x)| \leqslant \frac{M_2}{r^{\beta/2}} (t 2^{-l})^{\beta/2} < \frac{M_2}{r^{\beta/2}} 2^{-\beta/2} t^\beta \leqslant M_3 t^\beta.$$

To estimate $\| g_{1,t}(x)\|_{2r}$ we recall that $t < 1/16r$ implies that Supp $g_{1t}(x) \subset [0, \frac{7}{8}]$ and there it is enough to estimate sup $| x^r g_{1,t}^{(2r)}(x)|$ $(8^{-r} \leqslant (1-x)^r \leqslant 1)$. For $3 \cdot 4^{-m-1} \leqslant x \leqslant 4^{-m}$, $g_{1,t}(x) = f_{t2^{-m+1}}(x)$ and, using (5.6),

$$\begin{aligned} | x^r g_{1,t}^{(2r)}(x)| &= | x^r f_{t \cdot 2^{-m+1}}^{(2r)}(x)| \leqslant M_2 (t \cdot 2^{-m+1})^{-2r} x^r \cdot \frac{(t \cdot 2^{-m+1})^\beta}{x^{\beta/2}} \\ &\leqslant M_2 t^{-2r+\beta} 2^{-2r+\beta} \cdot 2^{m(2r-\beta)} (4^{-m})^{r-\beta/2} 4^{m(r-\beta/2)} \leqslant M_3 t^{-2r+\beta}. \end{aligned}$$

For $x \leqslant 4^{-l-1}$, $g_{1,t}(x) = f_{t2^{-l}}(x)$ and, using (5.6), we have

$$\begin{aligned} | x^r g_{1,t}^{(2r)}(x)| &\leqslant M x^r (t \cdot 2^{-l})^{-2r} (t \cdot 2^{-l})^{\beta/2} \leqslant M_4 4^{-lr} \cdot (2^{-l} 2^{-l})^{-2r} (2^{-2l})^{\beta/2} \\ &\leqslant M_4 (2^{-l})^\beta (2^{-l})^{-2r} \leqslant M_5 t^{-2r+\beta}. \end{aligned}$$

To show this inequality is valid for $4^{-m} < x < 3 \cdot 4^{-m}$ we will first define

$$\begin{aligned} f_{h_1,h_2}(x) = [f_{h_1}]_{h_2}(x) &= \sum_{j=1}^{2r} \sum_{i=1}^{2r} \binom{2r}{j} \binom{2r}{i} (-1)^{i+j} \left(\frac{2r}{h_1}\right)^{2r} \left(\frac{2r}{h_2}\right)^{2r} \\ &\times \int_0^{h_1/2r} \cdots \int_0^{h_1/2r} \int_0^{h_2/2r} \cdots \int_0^{h_2/2r} f(x + j(u_1 + \cdots + u_{2r}) \\ &+ i(v_1 + \cdots + v_{2r})) \times du_1 \cdots du_{2r} \, dv_1 \cdots dv_{2r} . \end{aligned} \tag{5.8}$$

Inspecting the definition, we observe that

$$f_{h_1,h_2}(x) = f_{h_2,h_1}(x). \tag{5.9}$$

Also we shall show

LEMMA 5.2. *Let $g(x) \in C^{2r}[x_0, 1]$ such that $| g^{(2r)}(x)| \leqslant M$ in $[x_0, 1]$, then*

$$\left|\left(\frac{d}{dx}\right)^k [g(x) - g_h(x)]\right| \leqslant M \cdot k \cdot h^{2r-k} \qquad \text{for} \quad x_0 \leqslant x < \frac{3}{4}.$$

Proof. We use Taylor's formula

$$\begin{aligned} g^k(x + \eta) &= g^{(k)}(x) + \eta g^{(k+1)}(x) + \cdots + \frac{\eta^{2r-k-1}}{(2r-k-1)!} g^{(2r-1)}(x) \\ &+ \frac{1}{(2r-k-1)!} \int_x^{x+\eta} (x + \eta - v)^{2r-k-1} g^{(2r)}(v) \, dv \end{aligned}$$

and we can recall that $\sum_{j=0}^{2r}\binom{2r}{j}(-1)^j(j\eta)^m = 0$ for $m < 2r$ which we use for $m \leqslant 2r - k - 1$ and therefore, using

$$\int_x^{x+\eta} |x+\eta-v|^{2r-k-1}\,|g^{(2r)}(v)|\,dv \leqslant M\frac{\eta^{2r-k}}{2r-k},$$

we have

$$\begin{aligned}&\left|\left(\frac{d}{dx}\right)^k [g(x)-g_h(x)]\right| \\ &\quad\leqslant \sum_{j=0}^{2r}\binom{2r}{j}\frac{M}{(2r-k)!}\int_0^{h/2r}\cdots\int_0^{h/2r}(j(u_1+\cdots+u_{2r}))^{2r-k}\,du_1\cdots du_{2r} \\ &\quad\leqslant KMh^{2r-k}.\end{aligned}$$

To prove now $|x^r g_t^{(2r)}(x)| \leqslant Mt^{-2r+k}$ for $4^{-m-1} < x < 3\cdot 4^{-m}$ we write

$$g_{1t}(x) = f_{t2^{-m}}(x)\,\psi_m(x) + f_{t2^{-m+1}}(x)(1-\psi_m(x))$$

since $1-\psi_{m+1}(x) = \psi_{m-1}(x) = 1$ for $4^{-m-1} < x < 3\cdot 4^{-m}$. We can also express $g_{1t}(x)$ by

$$\begin{aligned}g_{1t}(x) &= (f_{t2^{-m}}(x) - f_{t2^{-m},t2^{-m+1}}(x))\,\psi_m(x) + f_{t2^{-m},t2^{-m+1}}(x) \\ &\quad+ (f_{t2^{-m+1}}(x) - f_{t2^{-m},t2^{-m+1}}(x))(1-\psi_m(x)) \equiv I_1(t,x) + I_2(t,x) \\ &\quad+ I_3(t,x).\end{aligned}$$

The estimate of $I_2^{(2r)}(t,x)$ is given by

$$x^r\,|I_2^{(2)}(t,x)| \leqslant 2^{2r}x^r\,|f_{t2^{-m+1}}^{(2r)}(x)| \leqslant 2^{2r}M_3t^{-2r+\beta}.$$

To estimate $I_1(t,x)$ we recall $\psi\in C^{2r}$ and therefore $|\psi^{(k)}(x)| \leqslant K_1$ and $|\psi_m^{(k)}(x)| \leqslant 4^{mk}K_1$. Using the above and Lemma 5.2, we have

$$\begin{aligned}&x^r\,|I_1^{(2r)}(t,x)| \\ &\quad= x^r\left|\sum_{k=0}^{2r}\binom{2r}{k}\psi_m^{(k)}(x)\left(\frac{d}{dx}\right)^{2r-k}[f_{t2^{-m}}(x) - f_{t2^{-m},t2^{-m+1}}(x)]\right| \\ &\quad\leqslant 3^r(4^{-m})^r\sum_{k=0}^{2r}\binom{2r}{k}4^{mk}K_iK(t2^{-m+1})^k\,M_2(t2^{-m})^{-2r}\cdot(t2^{-m})^\beta\frac{1}{(4^{-m})^{\beta/2}} \\ &\quad\leqslant L4^{-m}t^{-2r+\beta}2^{2mk-mk}t^k2^{2mr} \\ &\quad\leqslant Lt^{-2r+\beta}\ (\text{since } t^k \leqslant (2^{-l})^k \leqslant 2^{-mk}).\end{aligned}$$

The estimate of $x^rI_3^{(2r)}(t,x)$ is similar.

REFERENCES

1. H. BERENS AND G. G. LORENTZ, Inverse theorems for Bernstein polynomials, *Indiana Univ. Math. J.* **21** (1972), 693–708.
2. P. L. BUTZER, Linear combinations of Bernstein polynomials, *Canad. J. Math.* **5** (1953), 559–567.
3. Z. DITZIAN AND C. P. MAY, A saturation result for combinations of Bernstein polynomials, *Tôhoku Math. J.* **28** (1976), 363–372.
4. G. G. LORENTZ, "Bernstein Polynomials," Univ. of Toronto Press, Toronto, 1953.
5. C. P. MAY, Saturation and inverse theorem for combinations of a class of exponential-type operators, *Canad. J. Math.* **28** (1976), 1224–1250.

Printed in Belgium